Grade 6 Math Circles
Week of 20th November
Matrices

Exercise Solutions

1. Putting this in a matrix, we have $A = \begin{bmatrix} 6 & 10 \\ 10 & 7 \end{bmatrix}$
2. The dimensions, starting from the left, are $(2 \times 1), (2 \times 3), (3 \times 3)$, and (2×2)
3. Computing the matrix expressions, we have
   (a) $\begin{bmatrix} 5 & 0 \\ 3 & 3 \end{bmatrix}$
   (b) $\begin{bmatrix} 2 & 5 \\ 7/2 & -2 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix}$
4. Computing the matrix expression, we have
   (a) $\begin{bmatrix} 20 & 5 \\ 20 & 10 \end{bmatrix}$
   (b) $\begin{bmatrix} 6 & -15 \\ 21/2 & -6 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
5. Swapping the rows and columns, we have that the transpose of the following matrices
   (a) $A^T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$
   (b) $A^T = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
   (c) $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

6. Using our formula for the area, we have that

   (a) $p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $q = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies \text{Area} = \left| \det \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) \right| = 3 \text{ units}^2$

   (b) $p = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$ and $q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \text{Area} = \left| \det \left(\begin{bmatrix} 1/2 & 1 \\ 3 & 0 \end{bmatrix} \right) \right| = 3 \text{ units}^2$

   (c) $p = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$ and $q = \begin{bmatrix} -1 \\ 1/5 \end{bmatrix} \implies \text{Area} = \left| \det \left(\begin{bmatrix} 10 & -1 \\ 3 & 1/5 \end{bmatrix} \right) \right| = 8 \text{ units}^2$

   (d) $p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \text{Area} = \left| \det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right| = 1 \text{ units}^2$

Problem Set Solutions

1. We know that for matrix addition and subtraction, the matrix dimensions must be the same, so we have that

   (a) $5A$ is possible, and has dimension (2×2)

   (b) $B + E^T$ is possible, and has dimension (2×3)

   (c) $C + 2E$ is not possible

   (d) $-3A + D^T$ is possible, and has dimension (2×2)

   (e) $A + 4B$ is not possible

2. Computing these matrix expressions, we have that

   (a) $5A - B^T = \begin{bmatrix} 0 & 17 \\ -10 & 14 \end{bmatrix}$

   (b) $3B + C = \begin{bmatrix} 19 & 31 \\ 11 & 4 \end{bmatrix}$

   (c) $A + (B - 2C) = \begin{bmatrix} -2 & 12 \\ -1 & 2 \end{bmatrix}$

   (d) $B - (2A + C) = \begin{bmatrix} -1 & 1 \\ 1 & -6 \end{bmatrix}$

3. Given our formula for the Trace, we have that

   (a) $\text{Tr}(A) = 2$

   (b) $\text{Tr}(A) = 11$

(c) $\text{Tr}(A) = 61/6$

4. Given the formula for the determinant of the (3×3) matrix, we can compute the areas and find that

   (a) Volume = 3 units^3

   (b) Volume = 14 units^3

   (c) Volume = 15 units^3

5. We know the formula for the area of the parallelogram given these vectors, but if we follow it, we find that

$$\left|\det\left(\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\right)\right| = 0$$

   It is equal to 0 because the two vectors are multiples of each other, that is $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, they don't even form a parallelogram to begin with. They are in a straight line on top of one another.

6. Computing the following matrix multiplications we have that

   (a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 8 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 10 & 0 \end{bmatrix}$

   You won't always get a square matrix, if you have two matrices, you can only multiply if the dimensions satisfy $(n \times m) \cdot (m \times k)$. The resulting matrix will have dimensions $(n \times k)$.

7. We know from the lesson the two matrices that rotate a vector by 90° and reflect across the line $y = x$. Thus, the resulting vector by rotating then reflecting is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Doing this with reflection first, rotation second, we get that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So order does matter!